## LITERATURE CITED

1. A. S. Pipkin, "Finite deformations of ideal fibered composites," Mekh. Kompozitn. Mater., No. 2, 138-143 (1978).
2. L. W. Morland, "Existence of solutions of plane traction problems for inextensible transversely isotropic elastic solids," J. Austral. Math. Soc., Ser. B, 19, No. 1 (1975).
3. Yu. A. Bogan, "On the second boundary value problem of elasticity theory for an essentially anisotropic plane with an elliptical hole," Prikl. Mekh., 17, No. 9 (1981).
4. S. G. Leknitskii, Anisotropic Plates [in Russian], Gostekhizdat, Moscow (1947).
5. B. A. Shoikhet, "On existence theorems in the linear theory of shells," Prikl. Mat. Mekh., 38, No. 3 (1974).
6. G. Duvaut and J.-L. Lions, Inequalities in Mechanics and Physics, Springer-Verlag, New York (1978).
7. R. Rostamian, "Internal constraints in linear elasticity," J. Elast., 11, No. 1 (1981).
8. Yu. A. Bogan, "Some singularly perturbed boundary value problems for a transversely isotropic elastic cylinder," in: Dynamics of a Deformable Solid, Vol. 66 [in Russian], Inst. of Geophy., Academy of Sciences of the USSR, Siberian Branch, Novosibirsk (1984).


# CONTACT PROBLEM OF THE THEORY OF ELASTICITY FOR PRESTRESSED BODIES WITH CRACKS

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Fatigue-test results often have a large scatter, which is generally related to a range of uncontrollable factors including the structure of an residual-stress distribution in the surface layers of the material, errors in assembly of the part, instability of the regime parameters and lubricant properties, etc. The effect of some of these factors on the performance of machine parts such as bearings was examined in [1]. There has been less study of the effect of residual stresses unavoidably created by some type of treatment (thermal, thermochemical, mechanical work-hardening, etc.) on the contact fatigue of materials. This topic has been investigated only by experimental method, and the available literature sources do not offer an unambiguous treatment of this subject. For example, in [2] (p. 227), the authors dispute that residual stresses have a significant effect on the fatigue of bearing steels. Several authors [3-7] hold that the retardation of fatigue is favorably influenced by compressive residual stresses and unfavorably influenced by tensile residual stresses. Other studies [8] indicate that compressive stresses are intolerable and that small tensile residual stresses are useful. Thus, the question of the usefulness and measurement of the effect of residual stresses on fatigue fracture remains unanswered.

Experimental studies were made in [9, 10] on the effect of shear stresses on contact fatigue. It was found that such stresses have an adverse effect on the fracture process.

Here we propose a mechanical model for the combined effect of normal and shearing contact stresses on fracture on the one hand and, on the other hand, the effect of residual stresses in the surface layers on fracture. The problem is examined in an elastic formulation and is reduced to a system of integral and integrodifferential equations with additional conditions in the form of equalities and inequalities. A solution is obtained by asymptotic methods. We determine the distribution of contact stresses and the stress intensity factors at the crack tips. An analysis is made of the effect of different levels of shearing contact stresses and residual stresses, as well as their sign (tensile or compressive), on the stress intensity factors. Numerical results are presented.

Thus, on the basis of analysis of the proposed model, it is possible to comparatively evaluate the effect of the above-mentioned factors on contact fatigue.

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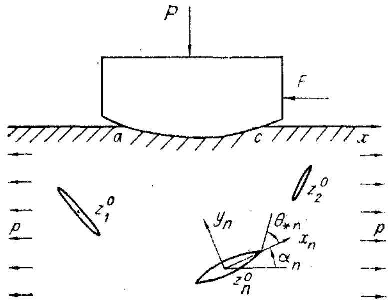


Fig. 1

1. Formulation of the Problem. We will examine a two-dimensional problem concerning the frictional interaction of a smooth die having a base z = f(x) with a prestressed elastic half-plane weakened by N rectilinear surface cracks. We will assume that friction on the boundary of the half-plane obeys Coulomb's law and that friction is absent on the edges of the crack. Preliminary (residual) stresses are created by compressive or tensile forces of constant intensity p applied at infinity (Fig. 1). Here, stresses [11] $p(1 - e^{-2i\alpha_n})/2$, induced by the stresses p, will be applied to the edges of the n-th crack ($\alpha_n$ is the angle between the x-axes of the n-th local coordinate system and the main coordinate system. Partial or complete superposition of the crack edges is permitted.

In dimensionless variables

$$\{x', \tau', a, c, x_k^{0'}, y_k^{0'}\} = \{x, \tau, x_i, x_e, x_k^0, y_k^0\}/b_0, \ \{q', p', p_k'\} =$$
$$= \{q, p, p_k\}/q_0, \ \{x_k', t'\} = \{x_k, t\}/l_k, \ \{v_k', u_k'\} = \{v_k, u_k\}/v_k^0,$$
$$f'(x') = \frac{\pi E'}{2P} f(x), \quad \delta^{0'} = \frac{\pi E'}{2P}\delta^0 - \ln\frac{1}{b_0}, \quad v_k^0 = \frac{4q_0 l_k}{E'}, \quad \delta_k = \frac{l_k}{b_0}$$

the problem is reduced to a system of singular integral and integrodifferential equations with additional conditions in the form of equalities and inequalities [11] (the primes have been omitted):

$$f(x) - \lambda\int_a^x q(t)\,dt + \frac{2}{\pi}\int_a^c q(t)\ln\frac{1}{|x-t|}\,dt -$$
$$-\frac{1}{\pi}\sum_{k=1}^N \delta_k \int_{-1}^1 \{v_k'(t) W_k^r(t, x) - u_k'(t) W_k^i(t, x)\}\,dt = \delta^0, \quad q(a) = q(c) = 0; \tag{1.1}$$

$$\int_a^c q(t)\,dt = \frac{\pi}{2}; \tag{1.2}$$

$$W_k(t, x) = ie^{-i\alpha_k}\frac{\overline{T}_k - T_k}{\overline{T}_k - x}, \ W_k^r = \operatorname{Re} W_k, \ W_k^i = \operatorname{Im} W_k; \tag{1.3}$$

$$\int_{-1}^1 \frac{v_n'(t)\,dt}{t - x_n} + \sum_{k=1}^N \delta_k \int_{-1}^1 \{v_k'(t) U_{nk}^r(t, x_n) - u_k'(t) V_{nk}^r(t, x_n)\}\,dt =$$
$$= \pi p_n(x_n) - \int_a^c q(t)\left[D_n^r(t, x_n) - \lambda G_n^r(t, x_n)\right]dt - \pi p\sin^2\alpha_n, \tag{1.4}$$
$$\int_{-1}^1 \frac{u_n'(t)\,dt}{t - x_n} + \sum_{k=1}^N \delta_k \int_{-1}^1 \{-u_k'(t) V_{nk}^i(t, x_n) + v_k'(t) U_{nk}^i(t, x_n)\}\,dt =$$
$$= -\int_a^c q(t)\left[D_n^i(t, x_n) - \lambda G_n^i(t, x_n)\right]dt - \pi\frac{p}{2}\sin 2\alpha_n;$$

$$U_{nk} = \overline{R_{nk} + S_{nk}}, \ V_{nk} = \overline{i(R_{nk} - S_{nk})}, \ U_{nk}^r = \operatorname{Re} U_{nk}, \ U_{nk}^i = \operatorname{Im} U_{nk},$$
$$V_{nk}^r = \operatorname{Re} V_{nk}, V_{nk}^i = \operatorname{Im} V_{nk}, D_n^r = \operatorname{Re}\overline{D}_n, D_n^i = \operatorname{Im}\overline{D}_n, G_n^r = \operatorname{Re}\overline{G}_n, G_n^i = \operatorname{Im}\overline{G}_n; \tag{1.5}$$

$$R_{nk}(t, x_n) = (1-\delta_{nk}) K_{nk}(t, x_n) + \frac{e^{i\alpha_k}}{2}\left\{\frac{1}{X_n - \overline{T}_k} + \frac{e^{-2i\alpha n}}{\overline{X}_n - T_k} + \right. \tag{1.6}$$

$$\left. + (\overline{T}_k - T_k)\left[\frac{1+e^{-2i\alpha_n}}{(\overline{X}_n - T_k)^2} - \frac{2e^{-2i\alpha_n}(X_n - T_k)}{(\overline{X}_n - T_k)^3}\right]\right\},$$

$$S_{nk}(t, x_n) = (1-\delta_{nk}) L_{nk}(t, x_n) + \frac{e^{-i\alpha_n}}{2}\left[\frac{T_k - \overline{T}_k}{(X_n - \overline{T}_k)^2} + \frac{1}{\overline{X}_n - T_k} - e^{-2i\alpha_n}\frac{X_n - T_k}{(\overline{X}_n - T_k)^2}\right],$$

$$K_{nk}(t, x_n) = \frac{e^{i\alpha_k}}{2}\left(\frac{1}{T_k - X_n} + \frac{e^{-2i\alpha_n}}{\overline{T}_k - \overline{X}_n}\right), \quad L_{nk}(t, x_n) =$$

$$= \frac{e^{-i\alpha_k}}{2}\left[\frac{1}{\overline{T}_k - \overline{X}_n} - \frac{T_k - X_n}{(\overline{T}_k - \overline{X}_n)^2} e^{-2i\alpha_n}\right],$$

$$D_n(t, x_n) = \frac{i}{2}\left[-\frac{1}{t - X_n} + \frac{1}{t - \overline{X}_n} - \frac{e^{-2i\alpha_n}(\overline{X}_n - X_n)}{(t - \overline{X}_n)^2}\right],$$

$$G_n(t, x_n) = \frac{1}{2}\left[\frac{1}{t - X_n} + \frac{1 - e^{-2i\alpha_n}}{t - \overline{X}_n} - e^{-2i\alpha_n}\frac{t - X_n}{(t - \overline{X}_n)^2}\right];$$

$$X_n = \delta_n x_n e^{i\alpha_n} + z_n^0, \; T_k = \delta_k t e^{i\alpha_k} + z_k^0, \; z_k^0 = x_k^0 + iy_k^0; \tag{1.7}$$

$$p_n(x_n) = 0, \; v_n(x_n) > 0; \; p_n(x_n) \leqslant 0, \; v_n(x_n) = 0; \qquad v_n(\pm 1) = u_n(\pm 1) = 0. \tag{1.8}$$

Here, x is the coordinate of a point of the contact region; $a$ and c are the coordinates of the boundaries of the contact region; $(x_k{}^0, y_k{}^0)$ and $\ell_k$ are the coordinates of the center and the half-length of the k-th crack; $x_k$ is the coordinate of a point in the local coordinate system connected with the k-th crack; $q = q(x)$ and $p_k = p_k(x_k)$ are the contact pressure and the stress acting on the edges of the k-th crack; $v_k = v_k(x_k)$ and $u_k = u_k(x_k)$ are the jumps in the normal and shear displacements of the edges of the k-th crack; f(x) is the form of the base of the die; $\lambda$ is the friction coefficient; $\delta^0$ is the convergence of the bodies; P is the force acting on the die; $q_0$ and $b_0$ are the characteristic pressure and half-width of the contact region ($q_0 b_0 = 2\pi^{-1}P$); $E' = E/(1 - \gamma^2)$ is the corrected elastic modulus of the material of the half-plane.

Thus, with assigned constants $z_k{}^0$, $\alpha_k$, $\delta_k$ (k = 1, 2, ..., N), $\lambda$, p and the function f(x), we need to use (1.1-1.8) to determine the constants $a$, c, and $\delta^0$ and the functions q(x), $v_k(x_k)$, $u_k(x_k)$ and $p_k(x_k)$ (k = 1, 2, ..., N). Having obtained the solution of the problem, it is easy to calculate the stress intensity factors for normal rupture $k_{1n}{}^{\pm}$ and shear $k_{2n}{}^{\pm}$ in dimensionless form

$$k_{1n}^{\pm} + ik_{2n}^{\pm} = \mp \lim_{x_n \to \pm 1} \sqrt{1 - x_n^2}\left[v_n'(x_n) + iu_n'(x_n)\right]. \tag{1.9}$$

It should be noted that at p = 0, problem (1.1)-(1.8) reduces to the problem studied in [12] in which preliminary stresses were absent.

The solution of problem (1.1)-(1.8) is a very complicated task due to the awkwardness of the equations and their kernels, the mutual effect of the contact stresses and the stress-strain state of the material near the cracks, and the need to determine the previously unknown boundaries of the contact region and superposed sections of the crack edges. It is evidently possible to study this problem in the general case only by numerical methods.

2. Asymptotic Investigation of the Problem. Certain simplifications can be made in the case when all of the cracks are small compared to the size of the contact region, i.e., when $\delta_0 = \max \delta_k \ll 1$.

It is interesting to examine that structure of the crack system in the elastic half-plane for which the distance between any two cracks is considerably greater than their dimensions, i.e.

$$z_n^0 - z_k^0 \gg \delta_0 \quad \forall n, k, n \neq k. \tag{2.1}$$

(The asymptotic relation $g \sim h$ means that $(g\bar{g})^{1/2} \sim (h\bar{h})^{1/2}$. Obviously, if $g \sim h$, then $\bar{g} \sim \bar{h}$. The asymptotic relations $g \gg h$ and $g \ll h$ are similarly determined.) Since the system of subsurface cracks belongs to the bottom half-plane, then $\operatorname{Im} z_n^0 \cdot \operatorname{Im} \bar{z}_k^0 < 0 \ \forall n, k$. Thus, it follows from (2.1) that

$$z_n^0 - \bar{z}_k^0 \gg \delta_0 \quad \forall n, k, n \neq k. \tag{2.2}$$

We will also assume that the cracks in the elastic half-plane lie at depths under the surface which are considerably greater than their dimensions:

$$z_n^0 - \bar{z}_n^0 \gg \delta_0 \quad \forall n. \tag{2.3}$$

The following estimates are obviously a consequence of (2.1)-(2.3)

$$\begin{gathered} T_k - X_n \gg \delta_0, \bar{T}_k - X_n \gg \delta_0 \ \forall n, \ k, n \neq k; \\ \bar{T}_k - T_k \gg \delta_0, x - T_k \gg \delta_0 \ \forall \ k, x. \end{gathered} \tag{2.4}$$

Having solved Eqs. (1.1) and (1.2) for q(x) and introducing the new independent variable $g = [2/(c - a)][x - (a - c)/2]$, we obtain [13]

$$q(y) = q^0(g, a, c) - \frac{\cos^2 \pi\gamma}{\pi^2 (c-a)} \times \tag{2.5}$$

$$\times R(g) \sum_{k=1}^{N} \delta_k \int_{-1}^{1} \left[ v_k'(\tau) \int_{-1}^{1} \frac{W_{k,t}^r(\tau, t)\, dt}{R(t)(t-g)} - u_k'(\tau) \int_{-1}^{1} \frac{W_{k,t}^i(\tau, t)\, dt}{R(t)(t-g)} \right] d\tau,$$

$$R(g) = (1+g)^{1/2-\gamma}(1-g)^{1/2+\gamma};$$

$$q^0(g, a, c) = f'(g) \frac{\sin 2\pi\gamma}{2(c-a)} + \frac{\cos^2 \pi\gamma}{\pi(c-a)} R(g) \int_{-1}^{1} \frac{f'(t)\, dt}{R(t)(t-g)}, \quad \operatorname{tg} \pi\gamma = \frac{\lambda}{2}; \tag{2.6}$$

$$\int_{-1}^{1} \frac{f_t'(t)\, dt}{R(t)} - \frac{1}{\pi} \sum_{k=1}^{N} \delta_k \int_{-1}^{1} \left[ v_k'(\tau) \times \int_{-1}^{1} \frac{W_{k,t}^r(\tau, t)\, dt}{R(t)} - u_k'(\tau) \int_{-1}^{1} \frac{W_{k,t}^i(\tau, t)\, dt}{R(t)} \right] d\tau = 0, \tag{2.7}$$

$$\int_{-1}^{1} \frac{t f_t'(t)\, dt}{R(t)} - \frac{1}{\pi} \sum_{k=1}^{N} \delta_k \int_{-1}^{1} \left[ v_k'(\tau) \int_{-1}^{1} \frac{t W_{k,t}^r(\tau, t)\, dt}{R(t)} - u_k'(\tau) \int_{-1}^{1} \frac{t W_{k,t}^i(\tau, t)\, dt}{R(t)} \right] d\tau = \pi.$$

It should be noted that $W_{k,t}^r(\tau, t) = \partial W_k^r(\tau, t)/\partial t$, etc. Meanwhile, the above-indicated substitution of variables was made in the kernels $W_k^r$ and $W_k^i$ in (2.5) and (2.7).

Analyzing the structure of the kernels $W_k$, $U_{nk}$, $V_{nk}$, $R_{nk}$, $S_{nk}$, $K_{nk}$, $L_{nk}$ and $D_n$ from (1.3) and (1.5)-(1.7), we can conclude that these kernels can be represented as asymptotic series in $\delta_n$ and $\delta_k$ which are regular for all x and t:

$$W_n(t, x) = \sum_{j=0}^{\infty} (\delta_n t)^j W_{nj}(x), \tag{2.8}$$

$$\{U_{nk}(t, x_n), V_{nk}(t, x_n)\} = \sum_{\substack{j+m=0 \\ j,m \geqslant 0}}^{\infty} (\delta_n x_n)^j (\delta_k t)^m \{U_{nkjm}, V_{nkjm}\},$$

$$\{D_n(t, x_n), G_n(t, x_n)\} = \sum_{j=0}^{\infty} (\delta_n x_n)^j \{D_{nj}(t), G_{nj}(t)\}.$$

Here, the quantities $U_{nkjm}$ and $V_{nkjm}$ do not depend on $\delta_n$, $\delta_k$, $x_n$, t, being functions of the constants $\alpha_n$, $\alpha_k$, $x_n^0$, $y_n^0$, $x_k^0$, and $y_k^0$. A similar dependence exists for $W_{kj}(x)$, $D_{kj}(t)$ and $G_{kj}(t)$.

Now let us proceed to the asymptotic solution of system (2.5)-(2.7), (1.3)-(1.8) at $\delta_0 \ll 1$. We will seek its solution by the method of regular perturbations [14] in the form of asymptotic expansions in powers of $\delta_0$:

$$\{q, a, c, v_n, u_n, p_n\} = \sum_{j=0}^{\infty} \delta_0^j \{q_j, a_j, c_j, v_{nj}, u_{nj}, p_{nj}\}. \tag{2.9}$$

After completing the solution of the problem, we can determine the constant $\delta^0$ from (1.1).

Let us perform an asymptotic analysis of Eqs. (2.5)-(2.7) to establish the effect of the contact stresses and overstresses on the stress-strain state of the elastic material near the cracks. We therefore limit ourselves to obtaining a binomial asymptotic solution. The mutual effect of the cracks on each other and on the contact stresses is described by terms of the order $O(\delta_0{}^2)$ and can be found in a similar manner.

Using (2.8) and (2.9) and having equated the coefficients with identical powers of $\delta_0$, we obtain

$$q_0(g) = q^0(g, a_0, c_0), \quad \int_{-1}^{1} \frac{f_t'(t)\,dt}{R(t)} = 0, \quad \int_{-1}^{1} \frac{tf_t'(t)\,dt}{R(t)} = \pi; \tag{2.10}$$

$$q_1(g) = 0,\ a_1 = c_1 = 0, \ldots \tag{2.11}$$

It should be noted that the constants $a_0$ and $c_0$ are determined from the last two equations of (2.10). Then the function $q_0(g)$ is calculated.

Now let us asymptotically analyze Eqs. (1.4)-(1.7). Having substituted the representations (2.8) and (2.9) into these equations, we find that

$$\int_{-1}^{1} \frac{v_{n0}'(t)\,dt}{t-x_n} = \pi p_{n0}(x_n) - \pi c_{n00}^r, \quad \int_{-1}^{1} \frac{u_{n0}'(t)\,dt}{t-x_n} = -\pi c_{n00}^i; \tag{2.12}$$

$$\int_{-1}^{1} \frac{v_{n1}'(t)\,dt}{t-x_n} = \pi p_{n1}(x_n) - \pi c_{n01}^r \frac{\delta_n}{\delta_0} x_n, \tag{2.13}$$

$$\int_{-1}^{1} \frac{u_{n1}'(t)\,dt}{t-x_n} = -\pi c_{n01}^i \frac{\delta_n}{\delta_0} x_n \ldots;$$

$$c_{nkj}^r + ic_{nkj}^i = \frac{1}{\pi}\int_{a_0}^{c_0} q_k(t)\,[\overline{D_{nj}(t)} - \lambda\overline{G_{nj}(t)}]\,dt + \tag{2.14}$$

$$+\, \delta_{k0}\delta_{j0}\frac{p}{2}\left(1 - e^{-2i\alpha_n}\right), \quad k, j = 0, 1,$$

Where $\delta_{ij}$ is the Kronecker symbol. We used Eqs. (2.11) in deriving Eqs. (2.12) and (2.13). Also, it is evident that the quantities $c_{njk}{}^r$ and $c_{njk}{}^i$ are independent of $x_n$.

For subsequent analysis of system (2.12)-(2.14), it is necessary to perform an asymptotic analysis of systems of alternative equalities and inequalities (1.8) at $\delta_0 \ll 1$. Having used representations (2.9), we have

$$\sum_{j=0}^{\infty} \delta_0^j p_{nj}(x_n) = 0, \quad \sum_{j=0}^{\infty} \delta_0^j v_{nj}(x_n) > 0, \tag{2.15}$$

$$\sum_{j=0}^{\infty} \delta_0^j p_{nj}(x_n) \leqslant 0, \quad \sum_{j=0}^{\infty} \delta_0^j v_{nj}(x_n) = 0.$$

We will assume that $v_{n0}(x_n) > 0$. Then it follows from the first condition of (2.15) at $\delta_0 \ll 1$ that $p_{nj}(x_n) = 0\ \forall j \geq 0$, while the sign of $v_{nj}(x_n)$ at $j \geq 1$ does not affect the satisfaction of the second inequality in (2.15). Now let us suppose the opposite, i.e., that $v_{n0}(x_n) = 0$. Then it is possible to realize one of two cases: a) $p_{n0}(x_n) < 0$; b) $p_{n0}(x_n) = 0$. In case "a," $p_n(x_n) < 0$ for $\delta_0 \ll 1$ outside the dependence on the values $p_{nj}(x_n)$ at $j \geq 1$. Here, from the last relation of (2.15) we find that $v_{nj}(x_n) = 0\ \forall j \geq 0$. In case "b," we find that $p_{n0}(x_n) = 0$ and $v_{n0}(x_n) = 0$ at $j \geq 1$ is now made during the next approximation at $\delta_0 \ll 1$. The approximation is done in a similar manner.

It follows from the last equations of (1.8) and (2.9) that

$$v_{nk}(\pm 1) = u_{nk}(\pm 1) = 0\ \forall n,\ k. \tag{2.16}$$

Let us examine the problem of the stress-strain state of the material near cracks in the zeroth approximation. It is described by Eqs. (2.12) together with system (2.16) and

$$p_{n0}(x_n) = 0,\ v_{n0}(x_n) > 0;\ p_{n0}(x_n) \leqslant 0,\ v_{n0}(x_n) = 0, \tag{2.17}$$

We will assume that $v_{n0}(x_n) > 0 \in x_n \in (-1, 1)$. Then from (2.17) $p_{n0}(x_n) = 0\ \forall\ X_n$ $(-1, 1)$, while from (2.12) we find [13] $v_{n0}(x_n) = c_{n00}{}^r\sqrt{1 - x_n{}^2} > 0$. Thus, $c_{n00}{}^r > 0$. It is easy to see that at $c_{n00}{}^r \le 0$, Eqs. (2.12) and (2.17) satisfy the functions $v_{n0}(x_n) = 0$, $p_{n0}(x_n) = c_{n00}{}^r \le 0\ \forall x_n \in (-1, 1)$. From the second equation of (2.12) and (2.16) we obtain [13] $u_{n0}(x_n) = c_{n00}{}^i\sqrt{1 - x_n{}^2}$. Thus, the solution of problems (2.12), (2.16), (2.17) has the form

$$v_{n0}(x_n) = c_{n00}^r\theta(c_{n00}^r)\sqrt{1 - x_n^2},\quad u_{n0}(x_n) = c_{n00}^i\sqrt{1 - x_n^2}, \tag{2.18}$$
$$p_{n0}(x_n) = c_{n00}^r\,\theta(-c_{n00}^r),$$

where $\theta(\cdot)$ is the Heaviside function.

System (2.15), together with (2.13) and (2.14) is subsequently analyzed similarly at $\delta_0 \ll 1$ with allowance for the expressions for the sought functions of the previous approximations. Here, the cracks have different configurations, depending on the values of the constants $c_{njk}{}^r$ and $c_{njk}{}^i$: completely open, partially closed, or completely closed.

It follows from analysis of system (2.15) that at $c_{n00}{}^r > 0$ (see (2.18)) $v_{n0}(x_n) > 0$ and $p_{nj}(x_n) = 0$, $x_n \in (-1, 1)\ \forall j \ge 0$, while the sign of $v_{nj}(x_n)\ \forall j \ge 1$ is immaterial. Thus, we obtain the following [13] from (2.13) by means of (2.16) at $c_{n00}{}^r > 0$:

$$v_{n1}(x_n) = \frac{\delta_n}{2\delta_0}c_{n01}^r x_n\sqrt{1 - x_n^2},\quad u_{n1}(x_n) = \frac{\delta_n}{2\delta_0}c_{n01}^i x_n\sqrt{1 - x_n^2}, \tag{2.19}$$
$$p_{n0}(x_n) = 0.$$

At $c_{n00}{}^r < 0$, it follows from analysis of system (2.15) (see (2.18)) that $p_{n0}(x_n) < 0$, $v_{nj}(x_n) = 0$, $x_n \in (-1, 1)\ \forall j \ge 0$, while the sign of $p_{nj}(x_n)$ at $j \ge 1$ is unimportant. Thus, from (2.13) at $c_{n00}{}^r < 0$

$$v_{n1}(x_n) = 0,\quad p_{n1}(x_n) = \frac{\delta_n}{\delta_0}c_{n01}^r x_n, \tag{2.20}$$

while the function $u_{n1}(x_n)$ is determined from (2.19).

We will examine the case $c_{n00}{}^r = 0$ when with zero approximation we have $v_{n0}(x_n) = p_{n0}(x_n) = 0$ $\forall x_n \in (-1, 1)$ (see (2.18). In this case the summation in (2.15) begins at j = 1. From (2.15) when $c_{n00}{}^r = 0$ we have an analog relationship (2.17).

$$p_{n1}(x_n) = 0,\ v_{n1}(x_n) > 0;\ p_{n1}(x_n) \leqslant 0,\ v_{n1}(x_n) = 0. \tag{2.21}$$

We will assume that $c_{n01}{}^r > 0$. Then on the basis of the form of the expression for $v_{n1}(x_n)$ in (2.19) obtained with the condition $p_{nj}(x_n) = 0\,\forall\ x_n \in (-1, 1)$, and the form of the right side of the first equation of (2.13), we can assume that the segment (−1, 1) occupied by the crack is subdivided into segments $(-1, b_{n1})$ and $(b_{n1}, 1)$ on which the relations $v_{n1}(x_n) = 0$ and $v_{n1}(x_n) > 0$ are satisfied. Here, from (2.13), (2.16), and (2.21) for $v_{n1}(x_n)$, $x_n \in (b_{n1}, 1)$

$$\int_{b_{n1}}^{1}\frac{v_{n1}'(t)\,dt}{t - x_n} = -\pi\frac{\delta_n}{\delta_0}c_{n01}^r x_n, \tag{2.22}$$

$$v_{n1}(b_{n1}) = v_{n1}(1) = 0. \tag{2.23}$$

The constant $b_{n1}$ is unknown and is determined from conditions (2.23) $p_{n1}(b_{n1}) = 0$, $|b_{n1}| \le 1$. Then from (2.2) we can easily determine the function $v_{n1}(x_n)$ [13], dependent on $b_{n1}$. After doing so, we use (2.13) with $x_n \in (-1, b_{n1})$ to determine the expression for $p_{n1}(x_n)$, which is also dependent on $b_{n1}$. Then having solved Eq. (2.23), we obtain $b_{n1} = -1/3$. As a result, $c_{n01}{}^r < 0$

$$v_{n1}(x_n) = \frac{\sqrt{3}\,\delta_n}{18\delta_0}c_{n01}^r(3x_n + 1)\sqrt{1 + 2x_n - 3x_n^2}\,\theta(1 + 2x_n - 3x_n^2), \tag{2.24}$$

$$p_{n1}(x_n) = \frac{\sqrt{3}\,\delta_n}{9\delta_0} c^r_{n01}(3x_n - 2)\sqrt{\frac{3x_n+1}{x_n-1}}\,\theta(3x_n^2 - 2x_n - 1).$$

We similarly obtain expressions for $v_{n1}(x_n)$ and $p_{n1}(x_n)$ at $c_{n00}{}^r = 0$ and $c_{n01}{}^r < 0$:

$$v_{n1}(x_n) = -\frac{\sqrt{3}\,\delta_n}{18\delta_0} c^r_{n01}(1 - 3x_n)\sqrt{1 - 2x_n - 3x_n^2}\,\theta(1 - 2x_n - 3x_n^2), \tag{2.25}$$

$$p_{n1}(x_n) = \frac{\sqrt{3}\,\delta_n}{9\delta_0} c^r_{n01}(3x_n + 2)\sqrt{\frac{3x_n-1}{x_n+1}}\,\theta(3x_n^2 + 2x_n - 1).$$

It should be noted that at $c_{n00}{}^r = 0$ and $c_{n01}{}^r > 0$ or $c_{n01}{}^r < 0$, the function $u_{n1}(x_n)$ is determined from (2.19).

Study of the case $c_{n00}{}^r = c_{n01}{}^r = 0$ leads to the need to examine terms of the asymptotic expansions having the order $O(\delta_0{}^2)$, which is outside the scope of the present investigation.

Using (1.9), (2.9), (2.18)-(2.20), (2.24) and (2.25), we can easily find expressions for the stress intensity factors

$$k^{\pm}_{1n} = c^r_{n00} \pm \frac{1}{2}\delta_n c^r_{n01} + \ldots, \quad c^r_{n00} > 0; \quad k^{\pm}_{1n} = 0, \quad c^r_{n00} < 0; \tag{2.26}$$

$$k^{\pm}_{1n} = \frac{\sqrt{3}\,\delta_n}{9} c^r_{n01}\left[\pm 7 - 3\theta(c^r_{n01})\right]\left[\frac{1 \pm \theta(c^r_{n01})}{1 \pm 3\theta(c^r_{n01})}\right]^{1/2} + \ldots, \quad c^r_{n00} = 0, \quad c^r_{n01} \neq 0;$$

$$k^{\pm}_{2n} = c^i_{n00} \pm \frac{1}{2}\delta_n c^i_{n01} + \ldots$$

3. Qualitative Analysis and Numerical Results. We will limit ourselves to examination of the problem for a parabolic die $f(x) = (x + d)^2$, $a = -b$, $c = b$, where d and b are previously unknown constants. Following [13], we obtain

$$q_0(x) = \cos\pi\gamma\,(b_0 + x)^{1/2-\gamma}(b_0 - x)^{1/2+\gamma}, \quad b_0 = (1 - 4\gamma^2)^{-1/2}, \tag{3.1}$$

$$d_0 = -2\gamma b_0, \quad \gamma = \frac{1}{\pi}\operatorname{arctg}\frac{\lambda}{\pi}.$$

Let $k_{1n}{}^{0\pm}$ and $k_{2n}{}^{0\pm}$ be the stress intensity factors in the absence of overstresses ($p = 0$). Then in the presence of overstresses p and a completely closed or completely open crack, we will have (see (1.16) from [12] and (2.14))

$$k^{\pm}_{1n} = k^{0\pm}_{1n} + p\sin^2\alpha_n, \quad k^{\pm}_{2n} = k^{0\pm}_{2n} + \frac{p}{2}\sin 2\alpha_n. \tag{3.2}$$

It follows from (3.2) that the overstress p does not affect $k_{1n}{}^{\pm}$ and $k_{2n}{}^{\pm}$ at $\alpha_n = 0$, while it exerts the maximum effect at $\alpha_n = \pi/2$ or $\pi/4$.

Curves 1 and 2 in Figs. 2 and 3 were obtained in the absence of preliminary stresses ($p = 0$) for $\lambda = 0.1$ and $0.2$, respectively, at (see Eq. (3.1)) $\gamma = 0.0159$, $b_0 = 1.0005$, $d_0 = -0.0318$ and $\gamma = 0.0317$, $b_0 = 1.002$, $d_0 = -0.0635$ ($y_n{}^0 = -0.2$, $\alpha_n = \pi/2$ and $\delta_n = 0.1$).

It follows from Fig. 2 that with an increase in $\lambda$ from 0.1 to 0.2, the stress intensity factor for normal rupture $k_{1n}{}^+$ in the nonprestressed half-plane increases by one order. It should also be noted that $k_{1n}{}^+(x_n{}^0)$ reaches a maximum next to the boundary of the contact region on the side opposite the slip of the die. This is due to the fact that it is in this region of the material that tensile stresses initiated by shear stresses $\tau$ in the contact are developed. The effect of the sign and the level of the prestresses in $k_{1n}{}^+$ is associated with an increase (tensile prestresses, $p > 0$) or a reduction (compressive prestresses, $p < 0$) in the size of the zone of tensile stresses in the surface layer and the level of these stresses. In fact, it follows from (3.2) that the coefficient $k_{1n}{}^+$ does not decrease with an increase in the tensile prestresses (curves 3, 4 correspond to $\lambda = 0.1$, $0.2$ and $p = 0.04$, $0.02$) for any values of $x_n{}^0$. Meanwhile, if $k_{1n}{}^+(p_1) > 0$, then at $p_2 > p_1$ $k_{1n}{}^+(p_2) > k_{1n}{}^+(p_1)$ (compare with curves 1 and 2). Similarly, it follows from (3.2) that $k_{1n}{}^+$ does not increase for any $x_n{}^0$ with an increase in the compressive prestresses (curves 5, 6 correspond to $\lambda = 0.1$, $0.2$ and $p = -0.01$, $-0.03$). Meanwhile, if $k_{1n}{}^+(p_1) > 0$, then at $p_2 < p_1$ $k_{1n}{}^+(p_2) < k_{1n}{}^+(p_1)$. The dependences of $k_{1n}{}^+$ on crack depth $y_n{}^0$ presented in [15] for $p = 0$ shows that crack growth proves to be possible only in a thin surface layer of the material of a thickness on the order of $b_0$. It follows from (3.2) that with tensile prestresses, the thickness of this layer increased. Meanwhile, beginning with a certain value $p > 0$, the growth of cracks

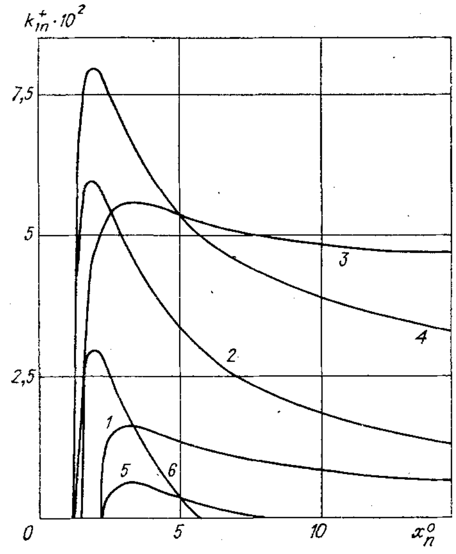


Fig. 2

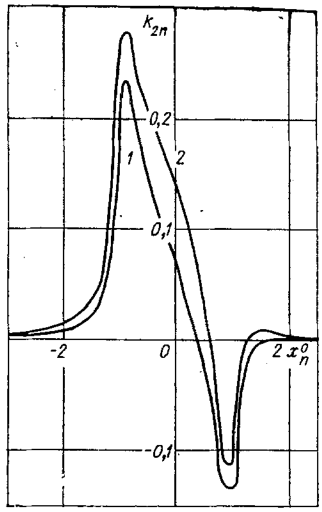


Fig. 3

is possible at any depth under the surface. In the case of compressive prestresses, the thickness of the layer decreases.

Figure 3 shows the relation for the shear stress intensity factor $k_{2n}^+$. It should be noted that at $\alpha_n = \pi/2$, we find from (3.2) that $k_{2n}^{\pm} = k_{2n}^{0\pm}$. Thus, the curves in Fig. 3 remain the same for any p. The value of $k_{2n}^+$ is heavily influenced by the orientation of the crack [12], while it is influenced very slightly by the friction coefficient $\lambda$. The relations $k_{2n}^+ = k_{2n}^+(x_n^0)$ reach extreme values next to the boundaries of the contact region. The behavior of $k_{2n}^+$ in relation to $y_n^0$ is different for different $x_n^0$ [15]. An analysis of Eqs. (2.26) shows that the coefficient $k_{2n}^+$ differs significantly from its limiting value $k_{2n}^+(y_n^0 = -\infty) = (p/2)\sin 2\alpha_n$ [see (3.2)] only in the thin surface layer of the material. It is evident that at $\delta_0 \ll 1$, $k_{1n}^+(k_{2n}^+)$ and $k_{1n}^-(k_{2n}^-)$ are close.

Such behavior of $k_{1n}$ can be used to create conditions that will discourage crack growth. It follows from [15] that, as a rule, cracks grow in a direction which deviates slightly from the perpendicular to the surface of the half-plane. Taking into account the fact that the quantity $k_{1n}$ is mostly responsible for crack growth [16], we find from (3.2) that at $y < y_n^0$ small cracks ($\delta_0 \ll 1$) do not grow (will be closed) when $p < p_0 = -\max\limits_{x_n^0} k_{1n}^{0\pm} + k_{th}$ ($k_{th}$ is the threshold value of $k_1$. When the threshold value is exceeded, the crack begins to grow [16]). Having put $k_{th} = 0$, we find from Fig. 2 that at $y < -0.2$, cracks do not grow if $p = p_0 = -0.016$ when $\lambda = 0.1$ and $p = p_0 = -0.06$ when $\lambda = 0.2$. A further increase in the compressive prestresses above $p_0$ leads to an increase in the crack resistance of layers of the material with $y > -0.2$ but does not affect the crack resistance of the underlying layers ($y \leq -0.2$). Thus, with a sufficiently high level of preliminary compressive stresses, the competing mechanism of surface fracture becomes important.

The results obtained here show that, in the elastic formulation, an increase in the friction coefficient and preliminary tensile stresses leads to intensification of the fracture process, while a reduction in the friction coefficient and an increase in preliminary compressive stresses leads to retardation of fracture.

## LITERATURE CITED


1. M. P. Kovalev and M. Z. Narodetskii, Design of High-Precision Ball Bearings [in Russian], Mashinostroenie, Moscow (1980).
2. A. G. Spektor, B. M. Zel'bet, and S. A. Kiseleva, Structure and Properties of Bearing Steels [in Russian], Metallurgiya, Moscow (1980).
3. R. L. Matson, "Fatigue, residual stresses, and surface-layer strengthening by work hardening," in: Fatigue of Metals [Russian translation], IL, Moscow (1961).
4. I. V. Kudryavtsev, Internal Stresses as a Strength Reserve in Machine Construction [in Russian], Mashgiz, Moscow (1951).

5. S. V. Serensen, "Fatigue resistance in connection with strain-hardening and design factors," in: Increasing the Fatigue Strength of Machine Parts by Surface Treatment [in Russian], GNTI (State Scientific and Technical Publishing House), Moscow (1952).
6. R. L. Scott, R. K. Kepple, and M. H. Miller, "The effect of processing-induced near-surface residual stress on ball bearing fatigue," in: Rolling Contact Phenomena, Elsevier Publishing Co., Amsterdam (1962).
7. I. Almen and P. Black, Residual Stress and Fatigue of Metals, McGraw-Hill, New York (1963).
8. I. M. Brozgol', Effect of Fine Finishing of the Roller Path on the Quality of Bearings (Survey), NIINavtoprom, Moscow (1973).
9. S. V. Pinegin, I. A. Shevelev, V. M. Gudchenko, et al., Effect of External Factors on Contact Strength During Rolling [in Russian], Nauka, Moscow (1972).
10. A. V. Orlov, O. N. Chermenskii, and V. M. Nesterov, Tests of Structural Materials for Contact Fatigue [in Russian], Mashinostroenie, Moscow (1980).
11. V. V. Panasyuk, M. P. Savruk and A. P. Datsyshin, Stress Distribution Around Cracks in Plates and Shells [in Russian], Nauk. Dumka, Kiev (1976).
12. I. I. Kudish, "Effect of shear stresses on the fatigue fracture of machine parts," Trenie Iznos, 7, No. 5 (1986).
13. I. Ya. Shtaerman, Contact Problem of the Theory of Elasticity [in Russian], Gostekhizdat, Moscow–Leningrad (1949).
14. M. Van Dyke, Perturbation Methods in Fluid Mechanics [Russian translation], Mir, Moscow (1967).
15. I. I. Kudish, "Study of the effect of subsurface defects in an elastic material on its fracture," in: Study, Calculation, and Design of Roller Bearings [in Russian], Spetsinformtsentr NPO VNIPP, Moscow (1986), No. 2.
16. S. Ya. Yarema, "Method of determining characteristics of the crack-growth resistance (fracture toughness) of materials under cyclic loading," Fiz. Khim. Mekh. Mater., No. 4 (1981).

# WAVE PROPAGATION IN CRUCIFORM ROD SYSTEMS

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It is often important to know the kind of vibrations of structural elements in vibration diagnostics problems of elastic structures. Transformation of the kind of vibrations is observed during propagation of vibrations in complex bifurcated structures and an element remote from the vibrations source can perform vibrations different from those which the source supplies.

Rods and plates are typical elements of elastic structures – consequently, a great deal of attention is usually paid to the study of vibrations propagating in rod and plate structures [1-3]. However, as a rule, rods and plates are studied in engineering theory approximations, which naturally constrains the frequency range of applicability of these models.

Theoretical and experimental investigations of the generation of longitudinal and bending waves through a cruciform connection of rods are carried out in this paper. The computation is performed by a nonclassical rod model [4]. This permits studying the wave processes in an elastic structure not only at low frequencies but also in the frequency range for which the lengths of the propagated waves become commensurate with the rod transverse dimensions.

1. A stiff cruciform connection of four rods is considered (Fig. 1). The propagation of longitudinal and bending waves in each of the rods is described by equations of the refined theory [4]

$$\rho_j S_j \frac{\partial^2 u_j}{\partial t^2} - E_j S_j \frac{\partial^2 u_j}{\partial x_j^2} - \rho_j \nu_j^2 I_{0j} \frac{\partial^4 u_j}{\partial x_j^2 \partial t^2} = 0,$$

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